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THE INVERSE SCATTERING PROBLEM FOR ACOUSTIC WAVES*

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All important decisions must be made on the basis of insufficient data - If You Meet the Buddha on the Road, Kill Him! by Sheldon Kopp

I. Introduction.

In this paper we shall survey recent progress and discuss open problems connected with the inverse scattering problem for acoustic waves. However before proceeding it is first necessary to be more precise on what we mean by "the inverse scattering problem" since this phrase has been used to describe a large variety of problems concerned with target identification in acoustic wave propagation. We first make a distinction between "scattering" and "diffraction" and note that the latter is basically a high frequency phenomena whereas the former is more accurately applied to low and intermediate values of the frequency. Hence in this paper we shall not discuss any of the important new results on the "inverse diffraction problem" but instead refer the reader to the recent paper of Brian Sleeman for a survey of these results ([19]). We shall further restrict our attention to the scattering of a plane time harmonic wave by a fixed bounded obstacle situated in a homogeneous medium and in particular to determine information

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about this obstacle from a knowledge of the asymptotic behavior of the scattered wave. Hence, we are excluding such topics as the scattering of waves by moving obstacles, the determination of the speed of sound in a non-homogeneous medium, and the location of equivalent sources. Finally, we shall only be concerned with determining two basic properties of the scattering obstacle, viz. its shape and/or its surface impedance. We note that although the problem of determining the shape of an obstacle from far field measurements has long been recognized as a basic problem in a variety of areas of technology such as radar, sonar, and tomography, the inverse impedance problem has received less attention. Nevertheless the inverse impedance problem is of basic importance in many applications since it gives information on the material composition of the unknown scattering object, e.g. in the case of sonar it can help answer the question of whether or not the scattering obstacle is a whale or a submarine.

The inverse scattering problem, as defined above, is particularly difficult to solve for two reasons: it is (1) nonlinear and (2) improperly posed. Of these two reasons it is perhaps the latter that creates the most difficulty. Indeed we shall see shortly that for a given measured far field pattern in general no solution exists to the inverse scattering problem, and if a solution does exist it does not depend continuously on the measured data. Hence before we can begin to construct a solution to the inverse scattering problem we must answer the question of what we mean by a "solution". At this point it is worthwhile recalling the remark of Lanczos: "A lack of information cannot be remedied by any mathematical trickery".

Hence in order to determine what we mean by a solution it is necessary to introduce "nonstandard" information that reflects the physical situation we are trying to model. Having resolved the question of what is meant by a solution, we then have to actually construct this solution, and this is complicated not only by the fact that the problem is nonlinear, but also the fact that the above mentioned "nonstandard" information has been incorporated into the mathematical model.

We shall now give a mathematical formulation of the inverse scattering problem and outline the specific topics we want to discuss in this paper. We shall formulate our problem in $\mathbb{R}^2$ and state, when appropriate, the necessary modifications that are needed to consider the full three dimensional problem. Let D be a bounded, simply connected domain in $\mathbb{R}^2$ with smooth boundary ∂D and unit outward normal ν. If we let $\lambda = \lambda(\underset{\sim}{x}) \geq 0$ denote the continuous surface impedance of ∂D and $k>0$ the wave number, then the impedance boundary value problem for acoustic waves can be mathematically formulated as the problem of finding a function $u \in C^2(\mathbb{R}^2 \backslash \bar{D}) \cap C^1(\mathbb{R}^2 \backslash D)$ such that

$$u = u^i + u^s \quad \text{in} \quad \mathbb{R}^2 \backslash D \tag{1.1a}$$

$$\Delta_2 u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D} \tag{1.1b}$$

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D \tag{1.1c}$$

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{1.1d}$$

where the "incoming wave" u^i is an entire solution of the Helmholtz equation (1.1b) and the "scattered wave" u^s satisfies the radiation condition (1.1d) uniformly with respect to θ where (r,θ) are polar coordinates. We shall also consider the case when λ is infinite, i.e. the boundary condition (1.1c) becomes

$$u = 0 \quad \text{on } \partial D. \tag{1.1c$'$}$$

In this case we shall refer to the scattering obstacle as being "soft". The existence and uniqueness of a solution to (1.1a)-(1.1d) and (1.1a), (1.1b), (1.1c$'$), (1.1d) is well known (c.f. [10]). We shall see shortly that if u is a solution of (1.1a)-(1.1d) (or (1.1a), (1.1b), (1.1c$'$), (1.1d) then u^s has the asymptotic behavior

$$u^s(r,\theta) = e^{ikr}\, r^{-1/2} F(\theta;k) + O(r^{-3/2}) \tag{1.2}$$

where F is known as the <u>far field pattern</u> corresponding to the given incoming wave u^i. The inverse scattering problems we shall discuss in this paper are (1) given a knowledge of u^i, F and D, find λ, or (2) given a knowledge of u^i, F and the boundary condition (1.1c$'$), find D. We shall be more precise on what we mean by a "knowledge" of F shortly, but as mentioned above we in general only know F from measurements which are by definition inexact and this fact makes both of the above inverse scattering problems improperly posed. The basic fact we begin with is that the existence of a unique solution to the direct scattering problem (1.1a)-(1.1d) or (1.1a), (1.1b), (1.1c$'$), (1.1d) by the method of integral equations (c.f. [10]) defines a (nonlinear) mapping

$\underset{\sim}{T}$ from D or λ to F. Hence from an abstract point of view we can state our tasks as follows:

(1) Determine the range of $\underset{\sim}{T}$ (denoted by $R(\underset{\sim}{T})$) as a subset of $L^2[0,2\pi]$.

(2) Establish the existence of $\underset{\sim}{T}^{-1}$ on $R(\underset{\sim}{T})$, i.e. show the uniqueness of the solution to the inverse scattering problem.

(3) Determine a subset $X \subset R(\underset{\sim}{T})$ and an operator $\hat{\underset{\sim}{T}}^{-1}$ defined on $L^2[0,2\pi]$ such that $\hat{\underset{\sim}{T}}^{-1} = \underset{\sim}{T}^{-1}$ on X and $\hat{\underset{\sim}{T}}^{-1}$ is continuous on $L^2[0,2\pi]$, i.e. stabilize the inverse scattering problem (In order to do this it is necessary to assume a priori information about D or λ).

(4) Give a constructive method for determining $\hat{\underset{\sim}{T}}^{-1}x, x \in X$.

We shall examine what is known about the above four problems in the following sections and in addition give directions that should be taken if further progress is to be achieved.

II. The Mapping $\underset{\sim}{T}$ and its Range.

As mentioned in the Introduction we can obtain the solution of (1.1a)-(1.1d) (or the corresponding problem with Dirichlet boundary condition (1.1c')) by the method of integral equations. In particular we can represent the solution u^s of (1.1a)-(1.1d) in the form of a modified single layer potential

$$u^s(\underset{\sim}{x}) = \frac{1}{\pi} \int_{\partial D} \phi(\underset{\sim}{y}) G(\underset{\sim}{x},\underset{\sim}{y})\, ds(\underset{\sim}{y}) \quad ; \quad \underset{\sim}{x} \in \mathbb{R}^2 \backslash \bar{D} \tag{2.1}$$

where G is an appropriately chosen fundamental solution ([22]) and ϕ is the unique solution of an integral equation of the form

$$\phi + \underset{\sim}{K}_1\phi + \lambda \underset{\sim}{K}_2\phi = \frac{\partial u^i}{\partial \nu} + \lambda u^i \tag{2.2}$$

where $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ are compact integral operators that are independent of λ and defined on $C(\partial D)$. The solution of the corresponding Dirichlet problem can be represented in the form of a double layer potential ([22])

$$u^s(\underset{\sim}{x}) = \frac{1}{\pi} \int_{\partial D} \psi(\underset{\sim}{y}) \frac{\partial}{\partial \nu(\underset{\sim}{y})} G(\underset{\sim}{x},\underset{\sim}{y}) ds(\underset{\sim}{y}) \quad ; \ \underset{\sim}{x} \in \mathbb{R}^2 \backslash \bar{D} \tag{2.3}$$

where ψ is the unique soluton of an integral equation of the form

$$\psi + \underset{\sim}{K}_3\psi = u^i \tag{2.4}$$

where $\underset{\sim}{K}_3$ is a compact integral operator defined on $C(\partial D)$. If in (2.1), (2.2) or (2.3), (2.4) we now let $r = |\underset{\sim}{x}|$ tend to infinity and use the asymptotic behavior of $G(\underset{\sim}{x},\underset{\sim}{y})$ we obtain the relationship (1.2) and the mapping $\underset{\sim}{T}:\lambda \to F$ or $\underset{\sim}{T}:\partial D \to F$. In particular such a calculation establishes the validity of the following theorem:

<u>Theorem 1</u>: The far field pattern is an entire function of θ and a continuous function of k for $k>0$.

The above analysis shows both the nonlinear nature of the operator $\underset{\sim}{T}$ as well as the fact that $R(\underset{\sim}{T}) \neq L^2[0,2\pi]$. Furthermore since it is not possible to determine the analyticity of a function from inexact measurements, Theorem 1 implies that in general for a given measured far field pattern no solution exists to the inverse scattering problem unless further assumptions are made.

From (2.2) or (2.4) it is clear that the operator $\underset{\sim}{T}$ depends on u^i, i.e. $\underset{\sim}{T}=\underset{\sim}{T}(u^i)$. We can therefore pose the following question: If u^i ranges over all entire solutions of the Helmholtz equations where λ or D is kept fixed, is the corresponding set of far field patterns dense in $L^2[0,2\pi]$, or more concisely, does $\overline{R(\underset{\sim}{T})}=L^2[0,2\pi]$. The following example shows that this is not true in general.

Example: Consider problem (1.1a), (1.1b), (1.1c'), (1.1d) when D is the unit disk. Then since u^i is an entire solution of the Helmholtz equation we can expand u^i in the form

$$u^i(r,\theta) = \sum_{n=0}^{\infty} J_n(kr)[a_n\cos n\theta + b_n \sin n\theta] \tag{2.5}$$

where J_n denotes Bessel's function and the series (2.5) is uniformly convergent on any compact subset of $\mathbb{R}^2$. Then for $r\geq 1$ we can expand u^s in the uniformly convergent series

$$u^s(r,\theta) = -\sum_{n=0}^{\infty} H_n^{(1)}(kr)\,\frac{J_n(k)}{H_n^{(1)}(k)}[a_n \cos n\theta + b_n \sin n\theta] \tag{2.6}$$

where $H_n^{(1)}$ denotes Hankel's function of the first kind and from the asymptotic behavior of Hankel's function we have that the far field pattern for u^s is given by

$$F(\theta;k)= e^{i\pi/4}\sqrt{\frac{2}{\pi k}}\sum_{n=0}^{\infty}\frac{(-i)^n J_n(k)}{H_n^{(1)}(k)}[a_n \cos n\theta + b_n \sin n\theta] \tag{2.7}$$

If k_o^2 is an eigenvalue of the interior Dirichlet problem then $J_n(k_o)=0$ for some integer $n=n_o$ and hence in this case $F(\theta;k_o)$ is orthogonal to $\cos n_o\theta$ and $\sin n_o\theta$ for all incident fields u^i. Hence the class of far field patterns for such values of k is not dense in $L^2[0,2\pi]$.

It is an open question to determine if similar examples are valid for arbitrary domains D. If such examples exist this would establish an interesting relationship between the far field patterns of exterior boundary value problems for the Helmholtz equation and the (interior) eigenvalue problem for Laplace's equation.

The validity of the above example is based on the fact that the set

$$\left.\begin{matrix} J_n(kr)\ \cos n\theta \\ J_n(kr)\ \sin n\theta \end{matrix}\right. ;\ n = 0,1,2,... \qquad (2.8$$

is incomplete in $L^2(\partial D)$ if k is eigenvalue of the interior Dirichlet problem. However it can be shown ([5]; a simpler proof can be based on the ideas of [13]) that the set

$$\left.\begin{matrix} (\frac{\partial}{\partial\nu} + ik\lambda)\ J_n(kr)\ \cos n\theta \\ (\frac{\partial}{\partial\nu} + ik\lambda)\ J_n(kr)\ \sin n\theta \end{matrix}\right. ;\ n=0,1,2,... \qquad (2.9)$$

is complete in $L^2(\partial D)$ for arbitrary bounded domains provided $0<\lambda<\infty$ where λ is a constant. This fact implies the following theorem ([5]):

<u>Theorem 2</u>: Let λ be a constant such that $0<\lambda<\infty$. Then $\overline{R(\underset{\sim}{T})} = L^2[0,2\pi]$.

A wealth of questions concerning $R(\underset{\sim}{T})$ remain unanswered and this provides a major mathematical challenge of basic importance to the inverse scattering problem. We have already mentioned one of these questions in connection with the Dirichlet problem. Another is the following: Determine the compact set of far field patterns in $L^2[0,2\pi]$ that corresponds to a fixed incident field and wave number, but with (constant) surface impedance lying in a given compact subset of the positive real axis. As opposed to the problem answered by Theorem 2, this problem is complicated by the fact that it is nonlinear. A similar problem can be posed for λ fixed but ∂D lying in a compact set of closed surfaces.

III. The Existence of $\underset{\sim}{T}^{-1}$.

As we have already mentioned in the Introduction, the existence of $\underset{\sim}{T}^{-1}$ is equivalent to establishing the uniqueness of solutions to the inverse scattering problem. We first consider the inverse scattering problem of determining D from a knowledge of u^i and F where we assume that the boundary data is given by (1.1c') ([17]).

Theorem 3: D is uniquely determined by a knowledge of the far field pattern F for all angles θ and k on any interval of the positive real axis.

Proof: Suppose there existed two obstacles D_1 and D_2 having the same far field pattern F. Consider first the case when $\bar{D}_1$ and $\bar{D}_2$ are disjoint. Then from the analyticity of solutions to the Helmholtz equation, F uniquely determines u^s outside a disk containing D_1 and D_2 in its interior and we can conclude by analytic continuation that u^s is an entire solution of the Helmholtz equation satisfying the radiation condition. But this implies that $u^s \equiv 0$ ([15]), i.e. from (1.1c') we have $u^i = 0$ on ∂D_1. Hence u^i is an eigenfunction of the Laplacian for an interval of k values and this is a contradiction since the set of eigenvalues for the Laplacian is discrete. A similar argument holds when

$D=\bar{D}_1\cap\bar{D}_2$ is nonempty, in particular by analytic continuation one can show that the Laplacian defined in $D_1\setminus\bar{D}$ or $D_2\setminus\bar{D}$ has a continuous spectrum, which is a contradiction.

We now consider the inverse scattering problem of determining λ from a knowledge of u^i, F and D, i.e. the boundary value problem is given by (1.1a)-(1.1d). The following theorem is due to Colton and Kirsch ([8]).

Theorem 4: λ is uniquely determined by a knowledge of the far field pattern F for all angles θ and any fixed positive value of the wave number k.

Proof: Let λ_1 and λ_2 be two solutions of the inverse scattering problem corresponding to the same far field pattern F. Then by analytic continuation we can conclude that $u_1=u_2$ and $\partial u_1/\partial\nu = \partial u_2/\partial\nu$ on ∂D where u_1 and u_2 are the solutions of (1.1a)-(1.1d) corresponding to λ_1 and λ_2 respectively. Then from the boundary condition (1.1c) we have

$$(\lambda_1-\lambda_2)u_1 = 0 \quad \text{on } \partial D. \tag{3.1}$$

We now note that u_1 is not identically zero in any neighborhood $S\subset\partial D$ since if this were the case (1.1c) implies that $\partial u_1/\partial\nu\equiv 0$ in S and hence by Holmgren's uniqueness theorem ([3]) $u_1\equiv 0$ in $\mathbb{R}^2\setminus D$. But this is a contradiction since $u_1=u_1^s+u_1^i$ and u_1^s satisfies the radiation condition but u_1^i does not. Therefore if $\underset{\sim}{x}\in\partial D$ there exists a sequence of points $\underset{\sim}{x}_n\to\underset{\sim}{x}$ such that $u_1(\underset{\sim}{x}_n)\neq 0$. From (3.1) we have that $\lambda_1(\underset{\sim}{x}_n)=\lambda_2(\underset{\sim}{x}_n)$ and since λ_1 and λ_2 are assumed to be continuous we have $\lambda_1(\underset{\sim}{x})=\lambda_2(\underset{\sim}{x})$. Since $\underset{\sim}{x}$ is an arbitrary point on ∂D this completes the proof of the theorem.

We note that from Theorem 1 and the identity theorem for analytic functions, it suffices in Theorem 3 and Theorem 4 to assume that F is only known for an interval of θ values contained in $[0,2\pi]$ instead of for all values of θ in $[0,2\pi]$.

An open problem of considerable interest is to determine the validity of the above theorems if instead of the far field pattern we are given the <u>scattering cross section</u> σ defined by

$$\sigma(k;\underset{\sim}{\alpha}) = \int_0^{2\pi} |F(\theta;k,\underset{\sim}{\alpha})|^2 d\theta \tag{3.2}$$

where $F(\theta;k,\underset{\sim}{\alpha})$ is the far field pattern corresponding to the incoming plane wave $u^i = \exp(ik\underset{\sim}{x}\cdot\underset{\sim}{\alpha})$.

IV. The Operator $\hat{\underset{\sim}{T}}^{-1}$.

As pointed out in the Introduction, the inverse scattering problems we are considering in this paper are improperly posed in the sense that in general no solution exists and if a solution does exist it does not depend continuously on the initial data. Hence a major task in a satisfactory treatment of these problems is to restore stability. We shall accomplish this by assuming extra a priori information is available concerning the unknown impedance or scattering obstacle such that it is possible to conclude that the exact far field pattern lies in a compact set $X \subset R(\underset{\sim}{T})$. We shall then show that it is possible to define an operator $\hat{\underset{\sim}{T}}^{-1}$ defined on $L^2[0,2\pi]$ such that $\hat{\underset{\sim}{T}}^{-1} = \underset{\sim}{T}^{-1}$ on X and $\hat{\underset{\sim}{T}}^{-1}$ is continuous on $L^2[0,2\pi]$, thus restoring the continuous dependence of λ or D on the far field pattern. At this point we would like to mention that the term "a priori information" is not meant to imply that such information is impossible to obtain from experimental data, but only that this (in general nonstandard) information is sufficient to stabilize the problem under consideration.

We first consider the problem of determining the impedance λ from a knowledge of the far field pattern F where the incident wave u^i and the scattering obstacle are assumed known ([8]). By using the compactness of the operators $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ in (22) it is possible to show that the mapping $\underset{\sim}{T}:\lambda \to F$ is continuous and in particular if ϕ is the solution of the integral equation (2.2) we can write $F=F(\theta;\phi,\lambda)$ where F is a continuous function of θ,ϕ and λ. Now let $F\in L^2[0,2\pi]$ be the measured far field pattern. Then we can reformulate our inverse scattering problem as an optimization problem, denoted by P_f, as follows: Minimize

$$C_f(\phi,\lambda) = \int_0^{2\pi} |F(\theta;\phi,\lambda)-f(\theta)|^2 d\theta \tag{4.1}$$

subject to ϕ being a solution of (2.2) and $\lambda\in U$ where U is a "control set" to be specified shortly. We first note the following facts:

(1) The set U contains the a priori information we are assuming about λ.

(2) If $\lambda^*\in U$ is a solution of the inverse scattering problem for a given far field pattern f and ϕ^* is the corresponding density, then (ϕ^*,λ^*) is a solution of P_f since $C_f(\phi^*,\lambda^*)=0$.

(3) If (ϕ^*,λ^*) is a solution of P_f and $C_f(\phi^*,\lambda^*)=0$ then λ^* is a solution of the inverse scattering problem. If $C_f(\phi^*,\lambda^*)>0$ then the inverse scattering problem is not solvable for $\lambda^*\in U$, but λ^* is a best approximation in the sense that $||F-f||_{L^2[0,2\pi]}$ is minimal.

We now define our control set U in such a way that P_f has a solution. In particular let

$$U = \{\lambda \in C^+(\partial D) : |\lambda(\underset{\sim}{x})| \le M_1, |\lambda(\underset{\sim}{x}) - \lambda(\underset{\sim}{y})| \le M_2 |\underset{\sim}{x} - \underset{\sim}{y}|^{\alpha}\}$$

where $C^+(\partial D)$ is the cone in $C(\partial D)$ consisting of all continuous functions λ defined on ∂D such that $\lambda \ge 0$, and M_1, M_2 and $\alpha, 0 < \alpha \le 1$, are fixed constants. The Arzela-Ascoli Theorem implies that U is compact in $C(\partial D)$ and since the functional C_f is continuous we have the following theorem:

<u>Theorem 5</u>: Let $f \in L^2[0,2\pi]$. The P_f has a solution.

In general the solution of P_f is not unique. Let $\Phi^*(f)$ be the set of all solutions (ϕ^*, λ^*) of P_f. Then the compactness of U and the continuity of $\underset{\sim}{T}$ implies the following result on continuous dependence:

<u>Theorem 6</u>: If $f_n \to f$ in $L^2[0,2\pi]$, $(\phi_n^*, \lambda_n^*) \in \Phi^*(f_n)$, then there exists a convergent subsequence of $\{(\phi_n^*, \lambda_n^*)\}$ and every limit point lies in $\Phi^*(f)$.

Note that if P_f is uniquely solvable, i.e. $\Phi^*(f)$ is a single ordered pair (ϕ^*, λ^*), then Theorem 6 simply says that $f \to (\phi^*, \lambda^*)$ is a continuous mapping from $L^2[0,2\pi]$ into $C(\partial D) x U$. In particular the operator $\hat{\underset{\sim}{T}}^{-1} : f \to (\phi^*, \lambda^*)$, $f \in \underset{\sim}{T}U = X$ satisfies the conditions on $\hat{\underset{\sim}{T}}^{-1}$ as set forth in the opening paragraph of this section. If P_f is not uniquely solvable then we must interpret $\hat{\underset{\sim}{T}}^{-1}$ as a set valued mapping and the criteria on $\hat{\underset{\sim}{T}}^{-1}$ as set out in this opening paragraph must be modified in an obvious manner.

For the extension of the above results to the case of the inverse scattering problem for electromagnetic waves see [12].

The inverse scattering problem of determining the shape of a "soft" scattering obstacle from a knowledge of the far field pattern can be handled in a similar manner as the inverse impedance problem provided we can define an appropriate compact family of surfaces U. This problem was discussed in [7] where the set U consisted of those domains D contained in a fixed circular annulus such that ∂D has a uniformly bounded Hölder continuous tangent. In the case of the three dimensional inverse scattering problem it was also necessary to assume that D was starlike with respect to the origin ([1]).

In closing we mention that although we have succeeded in using a priori information to stabilize the inverse scattering problem, we have made no statement on the <u>type</u> of continuity that results (i.e. Lipschitz, Hölder, logarithmic, etc.). This is an important problem that remains to be investigated since if the continuity is too weak there is little hope of using the optimization problem in a constructive fashion to actually determine λ or D. For a discussion of this point in the case of <u>linear</u> inverse scattering problems we refer the reader to [2].

V. Constructive Methods for Determining $\hat{\underline{T}}^{-1}x, x \in \underline{T}U$.

The previous section shows that the inverse scattering problems we are considering in this paper can be reformulated as constrained, nonlinear optimization problems. The numerical solution of these optimization problems is presently being investigated by A. Kirsch ([16]). Related approaches have been previously considered by A. Roger ([18]) and B. Sleeman ([19]). Needless to say there are many open problems connected with such a numerical approach, particularly in connection with the questions of stability and convergence.

The computational problems associated with computing $\hat{\underset{\sim}{T}}x^{-1}, x\in \underset{\sim}{T}U$, are considerably simplified if one has access to low frequency data. We first briefly consider the problem of determining the surface impedance of an obstacle from low frequency far field data ([6], [21]). Let F be the far field pattern arising from the addition of the scattered waves corresponding to two incoming plane waves striking the obstacle from opposite directions. If we expand F in its Fourier series

$$F(\theta;k) = \sum_{n=-\infty}^{\infty} a_n(k)e^{in\theta} \tag{5.1}$$

then the (weighted) low frequency limit of the coefficient $a_n(k)$ determine a sequence of numbers b_n from which we can define the harmonic function

$$h(\underset{\sim}{x}) = \sum_{n=-\infty}^{\infty} b_n r^{-|n|} e^{in\theta}. \tag{5.2}$$

It can be shown that the series (5.2) converges for $r \geq a$ where a is the radius of a disk containing the scattering obstacle in its interior. From [6] we now have that the unknown impedance λ is the solution of the integral equation of the first kind

$$h(\underset{\sim}{x}) = \int_{\partial D} \lambda(\underset{\sim}{y}) N(\underset{\sim}{x},\underset{\sim}{y}) ds(\underset{\sim}{y}) \quad ; \quad |\underset{\sim}{x}| = a \tag{5.3}$$

where N denotes the Neumann function for Laplace's equation in the exterior of D. Since F is assumed to be only approximately known, the computation of the coefficients b_n ($|n| \leq N$ for some positive integer N) must be done through the use of regularization procedures. In particular the left hand side of (5.3) is not known exactly and the problem of solving (5.3) is hence improperly posed. However stabilization can be achieved by assuming a priori that $\lambda \epsilon U$, in agreement with the results of the previous section.

A similar precedure can be used to determine the shape of a soft scattering obstacle from a knowledge of the far field pattern at low values of the frequency ([3], [9]). This method is based on relating the low frequency limit of the Fourier coefficients of the far field pattern to the coefficients of the Laurent expansion of the (unique) analytic function f that conformally maps the exterior of the unit disk onto the (unknown) scattering obstacle such that at infinity f has the Laurent expansion

$$f(w) = aw + a_o + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots \qquad (5.4)$$

where $a>0$ is the mapping radius. In particular if μ_n denotes the (weighted) low frequency limit of the Fourier coefficients then the μ_n are related to the a_n by a relation of the form

$$\mu_n = 2\pi n\, a^{n-1}\, a_{n-1} + \text{lower order coefficients}, \qquad (5.5)$$

for $n=1,2.,,,.$ where the mapping radius a can be determined from experimental data. Furthermore, if the coefficients $a_o, a_1, \ldots a_N$ are determined from (5.5) and f_N is defined by

$$f_N(w) = aw + a_o + \frac{a_1}{w} + \ldots + \frac{a_N}{w^N} \qquad (5.6)$$

then from the Area Theorem in geometric function theory it is possible to deduce the L^2-error estimate

$$\frac{1}{2\pi} \int_o^{2\pi} |f(e^{i\phi}) - f_N(e^{i\phi})|^2 d\phi \leq \frac{a^2}{N+1}. \qquad (5.7)$$

Note that from (5.5) small values of a can cause large errors in the computation of the coefficients a_n whereas large values of a imply that the error estimate (5.7) is large. Hence in order to achieve stability a must be bounded from above and below. This will be guaranteed if D is known a priori to be contained in a given annulus, in agreement with the results of the previous section. Note however that the error estimate in this case is with respect to the L^2 norm instead of the pointwise estimates of Section IV and hence it is not necessary to assume a priori that ∂D has a uniformly bounded Hölder continuous tangent. This suggests the problem of examining the stability of the inverse scattering problem where continuity is measured with respect to norms different from the maximum norm and determining the appropriate compact sets that are associated with these norms.

The above function-theoretic approach to the inverse scattering problem of determining the shape of the scattering obstacle from a knowledge of the far field pattern was first given by Colton ([4]) and Colton and Kleinman ([9]). Further developments of this method have subsequently been provided by Hariharan ([14]), Sleeman ([20]), and Smith ([21]). A partially successful attempt to extend this approach to the inverse scattering problem in $\mathbb{R}^3$ has been given by Colton and Kress ([11]). The problem in this case is that instead of conformal mapping methods one must rely on the use of level curves in potential theory and such an approach yields much weaker results.

The discussion provided in this section obviously only touches the surface of the computational problems inherent in constructing approximate solutions to the inverse scattering problem and it is hoped that the future will yield new developments and insights in this direction. Indeed the effective numerical solution of the inverse scattering problem is the basic open problem in the field and is intimately linked to all of the problems we have previously mentioned. Our hope is that this paper has indicated recent progress and possible new directions to a sufficient extent to encourage others to enter this important and fascinating area of research.

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